Influence of noise on crisis-induced intermittency

Marek Franaszek[1,*] and Leone Fronzoni[2]
[1] Institut für Festkörperforschung, Kernforschungsanlage Jülich, Forschungszentrum Jülich, D-5170 Jülich, Germany
[2] Institute of Physics, University of Pisa, Piazza Torricelli 2, Pisa, Italy
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The experimental study of crisis-induced intermittency in the asymmetric double-well Duffing oscillator shows that small additive noise can change drastically the properties of a dynamical system after the crisis. For example, the probability distributions inside the left and the right wells may be identical for the oscillator with added weak noise, contrary to the strongly asymmetric distributions observed in the system without noise. The large attractor created in the crisis may be decomposed onto a pair of repellers which are successors of two smaller attractors coexisting before the crisis. After the crisis, noise changes independently the mean lifetime of each repeller.

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The chaotic time evolution of a dynamical system can undergo a sudden change when a control parameter λ of the system exceeds the critical value λ_0. Such an event is called a crisis [1] and we can observe different kinds of behavior independent of a particular configuration in phase space for $\lambda < \lambda_0$. For example, if before the crisis the strange chaotic attractor coexists with another attractor, and for $\lambda = \lambda_0$ this chaotic attractor touches its basin boundary, then for λ slightly greater than λ_0 transient chaos may be observed [2, 3]. In this boundary crisis the attractor is destroyed and replaced by a strange repeller, which is characterized by positive escape rate κ and finite mean lifetime τ, $\kappa = 1/\tau$. A trajectory, which is initiated inside what had been the basin of the attractor before the crisis, is at the beginning attracted to the repeller and lives in its immediate neighborhood for a while. Then, the trajectory leaves this region forever and approaches the attractor that survived the crisis. Investigating a large number of trajectories started from homogeneously distributed initial points, we can determine the distribution of lifetimes on a given repeller. Usually, for large time t this distribution has an exponential form

$$G(t) = G_0 e^{-\kappa t} \ . \qquad (1)$$

Another kind of behavior may arise when two coexisting strange attractors touch at their mutual basin boundary. Then, for $\lambda > \lambda_0$, crisis-induced intermittency may be observed [4–6]. In this case, two attractors merge into one large attractor which governs the stationary evolution after the crisis. During this stationary evolution the trajectory jumps only very rarely between the regions corresponding to the two former attractors, and it spends most of the time confined to the first or to the second region. Thus, it is natural to decompose the large attractor appearing after the crisis into a pair of repellers which are the remnants of the two attractors that coexisted before the crisis. Crisis-induced intermittency is a particular case when the mean lifetimes of both repellers are long but one can easily generalize this concept to many coexisting repellers of arbitrary lifetimes. One long trajectory may enter and then escape the neighborhood of a particular repeller many times, and therefore this general type of behavior is called multitransient chaos [7].

Both phenomena, i.e., transient chaos and crisis-induced intermittency, may be observed clearly only for a control parameter λ sufficiently close to the critical value λ_0. Usually in such a critical range of the control parameter small external noise imposed on deterministic dynamics has a large influence. Noisy repellers were studied in computer experiments [8] and the main results of these investigations may be listed as follows: (i) in the presence of small noise the distribution of lifetimes has still the exponential form given by Eq. (1), but with escape rate κ strongly dependent on noise amplitude σ; (ii) the dependence $\kappa(\sigma)$ is not monotonic and for fixed control parameter there is a well defined noise level $\sigma^* > 0$ such that $\kappa(\sigma^*)$ is minimal (i.e., the corresponding mean lifetime $\tau(\sigma^*)$ is longest); (iii) external noise may either increase or shorten the mean lifetime, and the particular response of the system depends not on the noise amplitude σ or the control parameter λ separately, but it depends on the dimensionless parameter $\rho = \sigma/(\lambda - \lambda_0)$. These results were obtained from investigations of transient chaos where a single strange repeller coexists with a stable attractor. In the current paper we report on the results from an experimental study of crisis-induced intermittency in the presence of small external noise. The attractor created after the crisis may be decomposed into a pair of weakly repulsive repellers. Therefore, it is interesting to check if added noise changes independently each repeller or if the reactions of both repellers are somehow correlated. One can also expect that the changes of the repellers (which are the basic components of the large attractor) should be reflected in the properties of this attractor.

It should be stressed that we investigate a dynamical system which has already passed the crisis, and switch-

*Permanent address: Institute of Physics, Cracow Pedagogical University, Podchorążych 2, Kraków, Poland.

ing between two repellers may be observed even without external noise. This configuration is different from that one studied in Refs. [9–11] where the control parameter λ was less than λ_0, and the observed behavior was called noise-induced crisis. In that case there were two coexisting attractors, and only because of the nonzero intensity of noise could the trajectory jump from one basin of attraction to the other. However, it is interesting that the scaling of mean lifetimes found in the precrisis configuration

$$\tau(\sigma, \lambda) = \sigma^{-\gamma} g(|\lambda - \lambda_0|/\sigma) \tag{2}$$

seems to have a dependence on the control parameter λ similar to the postcrisis configuration, i.e., not directly via λ but via the dimensionless variable ρ. In Eq. (2) g denotes a nonuniversal function and γ is the critical exponent connected with the deterministic crisis. Particular forms of both relations are different because for fixed λ (after crisis) the dependence $\tau(\sigma)$ is not monotonic [8], while before the crisis the corresponding dependence introduced in Refs. [9–11] is monotonically increasing with decreasing noise level σ. In our experiment we kept the control parameter fixed ($\lambda > \lambda_0$) and the noise amplitude was changing in a given range. Thus, we were not interested in the question of how noise can modify the dependence of τ on the control parameter λ. Instead, we checked how the important properties of a dynamical system (for fixed control parameters) are sensitive to external noise.

In our experiment we used an analog simulator based on multiplier devices assembled in such a way as to simulate the Duffing oscillator

$$\ddot{x}(t) + \gamma\dot{x}(t) - \alpha x(t) + \beta x^3(t) = \lambda \sin(2\pi\nu t) + \psi + \xi(t) \ , \tag{3}$$

where a nonzero bias ψ ensures the asymmetry of the system and $\xi(t)$ stands for the Gaussian noise of intensity D and variance Σ, $\langle \xi(t)\xi(t+s)\rangle = 2D\delta(s)$.

We have made the circuit by means of a minimum component technique developed in our laboratory and widely described elsewhere [12, 13]. The main components of the circuit were two integrators connected, in a loop, with two multipliers. The multipliers allowed us to get a cubic term in the Duffing equation. The off-set of the first multiplier provides the bias term ψ. A noise generator and a sine generator applied to the input of the first integrator gave the stochastic fluctuation and the drive, respectively. In a real analog experiment it is not possible to get a strictly δ-correlated noise. We could only approximate this idealized noise by an experimental signal $u(t)$ such that

$$\langle u(t)u(t+s)\rangle = \frac{d}{\tau_0}\exp(-s/\tau_0) \ , \tag{4}$$

where d and τ_0 are the intensity and the correlation time of the experimental the noise. Such noise may be obtained by passing the noise of wide bandwidth through a linear filter

$$\dot{u}(t) = -1/\tau_0 u(t) + w(t) \ , \tag{5}$$

where $w(t)$ approximates the Gaussian white noise of the unit variance (the so called Wiener process). For $s \to 0$ we can get the connection with a directly measured quantity: the root mean square noise voltage $\sigma = \sqrt{\langle u^2(t)\rangle}$,

$$\sigma^2 = \frac{d}{\tau_0} \ . \tag{6}$$

In our experiment σ varied in the range between 0 and 9.5 mV, while the correlation time $\tau_0 = 4.7\ \mu$s. Comparing the cutoff frequency ν_{cut} of our noise generator with the resonance frequency ν_0 of the Duffing oscillator, at the minimum of the potential ($\nu_{\text{cut}} = 133\nu_0$), we can consider our generator as a source of white noise. We checked the stability and confidence of the described setup by an experimental test of the characteristic quantities of the system, such as the resonance frequency $\nu_0 = 300$ Hz, the half-width of the resonance curve $\Delta = 50$ Hz, and the positions of potential minima, $x_L = -3V$, $x_R = 3V$. Special attention was paid to test a symmetry of the system for the case of vanishing bias, $\psi = 0$. Other electronic parameters characterizing our analog simulator were the following: the voltage amplitude (peak to peak) of the forcing $V_{\text{p.p.}} = 0.28$ V and the frequency of the drive $\nu_{\text{expt}} = 303.3$ Hz. After a suitable time rescaling, the parameters of Eq. (3) corresponding to the experimental ones obtained the values: $\alpha = 1$, $\beta = 0.12$, $\gamma = 0.23$, $\lambda = 3.98V_{\text{p.p.}}$ (i.e., $\lambda = 1.11$), $\nu = 0.227$, $\psi = 2.56 \times 10^{-4}$ for large potential asymmetry, and $\psi = 6 \times 10^{-5}$ for the nearly symmetrical case, $\Sigma = 12\sigma$. Data acquisition was made by means of a digital oscilloscope (Data Precision model DATA 6000) with a sampling frequency $\nu_s = \nu_0$ for lower noise ($\sigma < 6$ mV) and $\nu_s = 3.3\nu_0$ for higher noise. It must be stressed that, besides the controlled noise of amplitude σ obtained from the generator, intrinsic experimental noise of nonzero amplitude σ_0 was also present. The total amplitude of noise, which disturbed the deterministic evolution of the investigated system, should therefore be written as $\sigma_{\text{tot}} = \sigma_0 + \sigma$ (in our experiment $\sigma_0 < 0.2$ mV and we can put $\sigma_{\text{tot}} \approx \sigma$).

Two kinds of measurements were performed. In the first series of experiments the marginal probability distributions $P(x)$ were recorded for different noise levels σ and for a few different values of bias. Examples of such distributions are shown in Figs. 1 and 2. The plots in Fig. 1 correspond to the nearly symmetrical case when the depth of both wells is approximately the same (bias $\psi = 6 \times 10^{-5}$). It is well visible that the noise of a relatively large amplitude does not influence significantly the probability distribution $P(x)$. The asymmetry ratio $\phi = P_R/P_L$, where $P_R(P_L)$ is the total area under the right (left) peak, is nearly independent of σ in the whole range of applied noise ϕ close to 1. On the other hand, Fig. 2 refers to the case of strong asymmetry when the left well is deeper than the right one (bias $\psi = 2.56 \times 10^{-4}$). Examples of four distributions $P(x)$ obtained for fixed system parameters and different noise levels are shown in Fig. 2. In this case the asymmetry ratio ϕ is systematically increasing with the increase of noise amplitude σ. It is evident that such important characteristics of the dynamical system as the joint probability distribution $p(x)$ are drastically changed

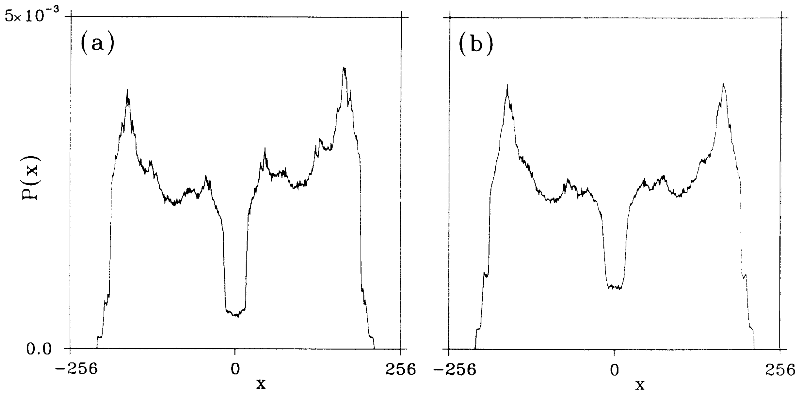

FIG. 1. Distribution $P(x)$ for the nearly symmetrical potential $V(x)$ (bias $\psi = 6 \times 10^{-5}$) and two different noise amplitude, σ: (a) $\sigma = 0$, asymmetry ratio $\phi = 1.11$; (b) $\sigma = 6.5$ mV, $\phi = 1.04$. For intermediate values of σ the plots and corresponding values of ϕ are similar. The bin size on the x axis is 1/512.

by noise. Consequently, the marginal probability $P(x)$ is also changed. Thus, all other mean quantities calculated with respect to this distribution must be affected by noise. We want to stress that for fixed system parameters and noise amplitude σ the changes caused by noise are stable in time. In particular, the shape of the distribution $P(x)$ and the asymmetry ratio ϕ may be easily repeated. Recording longer and longer time series we could observe that the corresponding distributions $P_t(x)$ were converging to the same stable asymptotic distribution $P_\infty(x) \equiv P(x)$.

In the second series of experiments the distributions of lifetimes inside the left and right well were determined. For each given system, parameters and noise amplitude σ, one very long trajectory was observed. The residence time inside a particular well was measured as the time interval between two successive transitions of $x(t)$ through zero. Thus, two sequences of time intervals were recorded and stored in the computer and then these sequences were used to produce the distribution of lifetimes inside the left and right well, respectively. In Fig. 3 examples of such distributions obtained for fixed system parameters and three different levels of noise are shown. More precisely, $G(t)$ denotes a probability that the trajectory does not leave a given well before time t. The common feature of all presented distributions is the existence of two different time scales. For small t a very fast decay dominates, while for $t > 0.03$ s, much slower decay is observed. Straight line fits are performed only for the flatter parts of the plots, and whenever we consider the escape rate κ or mean lifetime τ we have in mind the parameters obtained from these fits. The slow decay is connected with parts of time evolution during which the trajectory is confined to one particular well. Thus, the estimated pair of escape rates κ characterizes a pair of

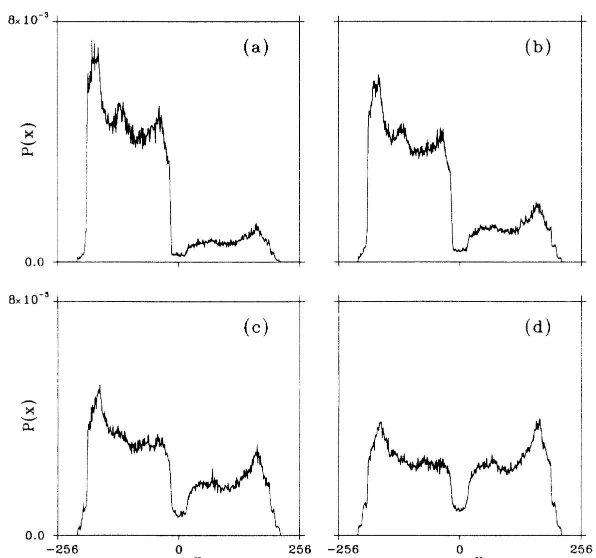

FIG. 2. Distribution $P(x)$ for the asymmetric potential $V(x)$ (bias $\psi = 2.56 \times 10^{-4}$) and different noise amplitudes σ: (a) $\sigma = 0$, asymmetry ratio $\phi = 0.16$; (b) $\sigma = 1.4$ mV, $\phi = 0.30$; (c) $\sigma = 4.2$ mV, $\phi = 0.56$; (d) $\sigma = 7$ mV, $\phi = 0.96$.

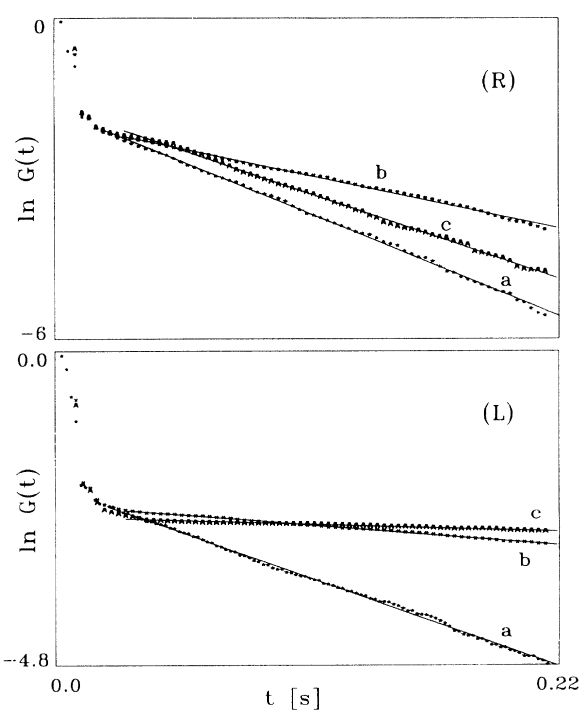

FIG. 3. Distribution of lifetimes $G(t)$ inside the left (L) and right (R) wells for fixed system parameters (the same as in Fig. 2) and different noise amplitudes: (a) $\sigma = 9.4$ mV; (b) $\sigma = 2.4$ mV; (c) $\sigma = 0$.

repellers which are the remnants of the attractors coexisting before the crisis. The steep parts of the plots in Fig. 3 are connected with a short lasting cycling of the trajectory above the barrier which separates two wells. A similar superposition of two individual chaotic transients of different mean lifetimes was observed in the numerical investigations of the Duffing oscillator [14]. It is evident that applied noise has no remarkable influence on the fast decay. On the contrary, the escape rate κ of slow decay is strongly noise dependent. In Fig. 4 the dependence of mean lifetime τ versus noise amplitude σ is shown. The system parameters were fixed at values giving large asymmetry of the potential $V(x)$ [notice the change of scale in Figs. 3 and Fig. 4 for the left (L) and right (R) repeller]. Therefore, the shape of the $\tau(\sigma)$ dependence for the left repeller may be different from the shape of the $\tau(\sigma)$ dependence for the right repeller. It should be also noticed that the relative change of mean lifetime is much larger for the left repeller than for the right one. Thus, the influence of noise on the repellers A_R and A_L is different. This interesting lack of correlation between both repellers is a natural consequence of the fact that each of them evolves in a different way when control parameter is passing the critical value. In the current analog experiments we were not able to observe the evolution of coexisting attractors for increasing the control parameter λ in a systematic way. However, numerical solutions of Eq. (3) (without noise and for other control parameters close to the experimental ones) suggest the following sequence of events [15]. For $\lambda < \lambda_0^{(R)}$ two different attractors A_L and A_R coexist. For $\lambda = \lambda_0^{(R)}$ the right attractor A_R collides with the basin boundary, and for $\lambda_0^{(R)} < \lambda < \lambda_0^{(L)}$ we can observe transient chaos. The final state is then connected with the left attractor A_L, but this state may be reached through two different transients. The shorter one can be seen when a starting point is chosen inside the existing for $\lambda < \lambda_0^{(R)}$ basin of attractor A_L. The longer transient is connected with the repeller R_R just born at $\lambda = \lambda_0^{(R)}$. This transient may be observed for starting points placed inside the basin of attraction corresponding to the attractor A_R which existed for $\lambda < \lambda_0^{(R)}$. For $\lambda = \lambda_0^{(L)}$ the left attractor A_L also collides with the unstable orbit, and for $\lambda > \lambda_0^{(L)}$ we can observe crisis-induced intermittency with the characteristic jumps between R_L and R_R. Basing on that we expect a similar evolution in our analog experiment.

Due to the strong asymmetry of the potential $V(x)$, the right well is shallower than the left one. If the results of Ref. [8] may be applied independently to the repellers R_L and R_R, then we should introduce two different dimensionless parameters $\rho_L = \sigma/(\lambda - \lambda_0^{(L)})$ and $\rho_R = \sigma/(\lambda - \lambda_0^{(R)})$, which link unambiguously the reaction of a particular repeller with the values of noise amplitude and control parameter. Therefore the same noise of amplitude σ may have a different influence on each repeller. However, we are not able to give the exact values of ρ_L and ρ_R because the critical parameters $\lambda_0^{(L)}$ and $\lambda_0^{(R)}$ could not be determined with satisfactory accuracy in the analog experiment. Therefore, the amplitude of applied noise σ instead of dimensionless parameter ρ is shown on the horizontal axis in Fig. 4, and we are not able to overlay two plots from Fig. 4 corresponding to the left (L) and the right (R) repellers on one common plot (as it was done in Ref. [11]). In order to verify systematically the scaling hypothesis for the currently discussed crisis-induced intermittency with noise, one should vary not only the noise level σ but also the control parameter λ (which was fixed in our experiment). Different shapes of dependence $\tau(\lambda, \sigma)$ shown in Fig. 4 do not contradict the scaling hypothesis because they may correspond to two different parts of the same plot $\tau(\rho)$ determined for two different ranges of dimensionless parameter ρ.

In the computer experiments it was shown [8] that for $\rho < 2.5$ noise can stabilize transient chaos, while for $\rho > 2.5$ the estimated mean lifetime is always shorter than in the noiseless case. In a real experiment the control of system parameters is more difficult and we are forced to achieve a compromise between two opposite tendencies. When we want to see transient chaos which is clearly elongated by noise, we should deal with a repeller of very long mean lifetime. It means that the control parameter λ should be very close to the critical value λ_0. But then the range of noise amplitude σ, where elongated transients may be observed, is very small and may be below the level of intrinsic noise σ_0, which is always present in a real experiment. Therefore elongated transient should be observed for λ sufficiently far from λ_0. In practice, we had to find an intermediate value of λ which was sufficiently large with respect to positive σ_0 and simultaneously sufficiently close to λ_0 in order to ensure the long mean lifetime. Thus, it is not surprising that we observe a monotonically decreasing dependence of the mean lifetime $\tau(\sigma)$ for the left repeller and an elongated transient for the right one ($\lambda_0^{(R)} < \lambda_0^{(L)}$ and therefore for

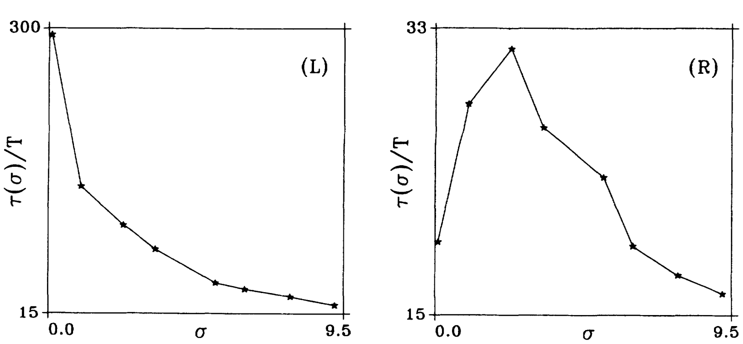

FIG. 4. Dependence of the mean lifetime τ connected with the left (L) and right (R) repeller versus noise level σ (in mV) for fixed system parameters (in the experimental circuit the period of the driving force $T = 3$ ms).

a fixed σ, $\rho_R < \rho_L$).

One remark should be added here. The nonmonotonic reaction of the deterministic system on external noise and the existence of specified noise amplitude σ^*, for which transient chaos possesses the longest mean lifetime, have some similarities with stochastic resonance [16, 17]. In this phenomenon many important characteristics (e.g., signal to noise ratio) are maximized for a certain resonant noise intensity. However, we must stress that stochastic resonance is observed for completely different parameters, namely, large noise intensity and very small amplitude of driving force. This amplitude is so small that a purely deterministic trajectory without the help of noise cannot jump over the barrier separating two wells of the potential. On the other hand, the intensity of the noise is so large that even without an external driving force the trajectory may switch between both wells in a purely stochastic way. In the current study of noisy crisis-induced intermittency the proportions are exactly reversed. For example, noise to drive the ratio Σ/λ is equal to 6.5% for the values of parameters corresponding to Fig. 2(d), while the mean unmodulated barrier height $\Delta V = (\Delta V_R + \Delta V_L)/2$ is equal to 2.19 in comparison with the drive amplitude $\lambda = 1.11$. Here, $\Delta V_{R,L} = |V(x_{R,L}) - V(x_0)|$ and x_R, x_L, and x_0 are the local extremes of the asymmetric double-well potential $V(x)$. In this case the amplitude of the driving force is sufficiently large to enable hopping between two wells without the assistance of stochastic perturbation. On the other hand, the noise intensity is so small that without external force there are no jumps over the barrier. Although stochastic resonance and noisy intermittency are observed for quite different ranges of parameters, they nevertheless have some common features [17]. The fact that noise of increasing intensity can wash out asymmetry of the potential $V(x)$ would not be surprising in the limit of large noise amplitude. However, the results shown in Fig. 2 were obtained in the limit of weak noise and therefore they should be explained as an unusually large sensitivity of the deterministic system on a small stochastic perturbation.

Before the final conclusions we want to add a comment about the decomposition of the large postcrisis attractor onto the pair of repellers R_L and R_R. Obviously, this procedure is not completed because the entire attractor includes not only orbits which are localized on a particular repeller but also long orbits which link both repellers. Moreover, besides the pair of repellers R_L and R_R there exists another strongly repulsive repeller R_0 which manifests its existence during the initial fast decay shown in Fig. 3. Recently, a similar critical configuration was investigated [18] in the logistic map in terms of the thermodynamical formalism. It appears that three main components of the entire attractor: R_L, R_0, and R_R play different roles in the description of the attractor. As far as we are interested in the geometrical properties, such as the capacity or the correlation dimension, the pair of repellers R_L and R_R approximates quite well the corresponding characteristics of the large attractor. However, when we are interested in the dynamical properties, such as the Lyapunov exponent or the metric entropy, the contribution of the repeller R_0 is predominant. Numerical experiments suggest that in the Duffing oscillator the configuration may be even more complicated, and we can observe crisis-induced intermittency between three as well as between four coexisting repellers (see Figs. 3 and 4 in Ref. [14]). In the current analog experiment we were interested in the geometrical properties of the large postcrisis attractor; for example, in the influence of small noise on the probability distribution $P(x)$. Therefore, we could decompose the entire attractor onto the pair of long-lived repellers R_L and R_R.

In the final conclusions we want to stress once again the exceptionally important role of small noise in the investigations of crisis-induced intermittency. When the control parameter λ of the investigated system is close to the critical value λ_0, the level of noise must be known and taken into account as one of the system parameters. The study of noise-induced intermittency [9–11] (for $\lambda < \lambda_0$) and crisis-induced intermittency with added noise ($\lambda > \lambda_0$), as discussed here, indicates clearly that a proper comparison between two similar experiments requires the use of properly rescaled system parameters. Existing after the crisis the large attractor has very unusual, strongly noise dependent properties. Such sensitivity of the dynamical invariants on noise amplitude is not known for other chaotic attractors, which exist sufficiently far from crisis. Therefore, the concept of multitransient chaos seems to be more suitable in the analysis of system behavior for λ close to λ_0. According to this approach we can analyze the influence of noise on each repeller independently. In particular, we can normalize probability distribution separately for the right and the left repeller. Then the obtained two different distributions $P_L(x)$ and $P_R(x)$ depend only weakly on noise level.

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